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General Lin-Maldacena solutions and PWMM instantons from supergravity

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Abstract: We use the Lin-Maldacena prescription to demonstrate how to find the supergravity solutions dual to arbitrary vacua of the plane wave matrix model and maximally supersymmetric Yang-Mills theory on $R \times S^2$, by solving the auxiliary electrostatics problem. We then apply the technique to study instantons at strong coupling in the matrix model.

Keywords: Gauge-gravity correspondence, M(atrix) Theories, Solitons Monopoles and Instantons.

Contents

1. Introduction

The AdS/CFT correspondence [1] has proven to be a remarkably useful tool for studying $\mathcal{N} = 4$ SYM at strong coupling. However, this is only the most well-known example of gauge/gravity duality. Lin and Maldacena [2], building on the work of [3], have found another very interesting class of dualities. This class relates SU(2|4) supersymmetric gauge theories (the plane wave matrix model, and maximally supersymmetric Yang-Mills theories on $R \times S^2$ and $R \times S^3/Z_k$) to type IIA string theory in various backgrounds. The metric, dilaton, and form fields in the supergravity solutions in question can be expressed in terms of a single function that is an axisymmetric solution of the Laplace equation in three dimensions. The gravity solutions, therefore, are specified by axisymmetric electrostatics configurations. The field theories in question have many vacua, and Lin and Maldacena [2] were able to determine the correspondence between the vacua and the supergravity solutions. These present interesting examples of the gauge/gravity correspondence as field theories living in different numbers of dimensions are dual to string theory in backgrounds that share similar features in the infrared region, such as throats with NS5- or D2-brane flux.

The electrostatics configurations corresponding to the field theory vacua are given by different arrangements of charged conducting disks [2]. For a general vacuum on the field theory side, and therefore a general disk configuration on the gravity side, solving the electrostatics problem is quite challenging, and explicit solutions are known in only some special cases.

One particular solution given by Lin and Maldacena [2] corresponds to two infinitely large disks held at fixed separation. They gave an explicit form for the gravity solution and argued that it should be dual to little string theory on S^5. The gravity dual was used in [4] to argue that little string theory on S^5 has interesting features that differ from the theory

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in flat space. An explicit solution has also been given in the case of a single isolated disk, dual to a vacuum of the maximally supersymmetric Yang-Mills theory on $R \times S^2$ [2, 5]. Also, in the region very close to the tip of a disk, the problem becomes two dimensional, and it is possible to solve it by conformal mapping [2]. More general explicit solutions, however, are not known.

In general this has prevented this set of dualities from being used to study the SU(2|4) symmetric field theories at strong coupling. It is interesting to consider what questions can be addressed from the information we do know on the gravity side. Recently, Lin [6] has made some progress in applying this correspondence to instanton calculations in the plane wave matrix model and maximally supersymmetric Yang-Mills theory on $R \times S^2$. In the case of the matrix model, this question had been studied directly in the field theory [7]. Lin gave explicit results for weak coupling from the supergravity side, and found precise agreement with the gauge theory analysis.

It would certainly be desirable to be able to perform other gauge theory calculations using the dual gravity description. It is, therefore, quite interesting to obtain more general supergravity solutions that would allow this to be done.

In this paper we will demonstrate that it is possible to reduce the generic electrostatics problem to a simple linear system that can be solved very simply using numerical methods. We will then use this technique to find some explicit results using Lin's prescription for instanton calculations on the dual gravity side. For a simple example electrostatics configuration, dual to a vacuum of the plane wave matrix model, we will give an explicit expression for the superpotential at strong coupling, and also the leading correction to Lin's result at weak coupling.

2. Supergravity solutions

In this section we will review the Lin-Maldacena formulation of supergravity solutions in terms of electrostatics problems; full details can be found in [2]. Then in subsections 2.1 and 2.2 we will discuss the solution of these problems.

To find the supergravity duals to field theories with SU(2|4) symmetry, Lin and Maldacena looked for similarly symmetric supergravity solutions. In particular, the bosonic part of this symmetry group is $R \times \mathrm{SO}(3) \times \mathrm{SO}(6)$ so the supergravity solutions should contain an S^2 and an S^5. Interestingly, with this restriction all of the supergravity fields can be expressed in terms of a single function of the two remaining coordinates. For the supergravity equations to be satisfied, this function must be an axisymmetric solution to the Laplace equation in three dimensions.

The full supergravity solution in terms of this function, in the string frame, is

$$\begin{aligned} ds^2 &= \left(\frac{\ddot{V} - 2\dot{V}}{-V''}\right)^{1/2} \left(-\frac{4\ddot{V}}{\ddot{V} - 2\dot{V}}dt^2 + \frac{-2V''}{\dot{V}}(dr^2 + dz^2) + 4d\Omega_5^2 + 2\frac{V''\dot{V}}{\Delta}d\Omega_2^2\right) \\ e^{4\Phi} &= \frac{4(\ddot{V} - 2\dot{V})^3}{-V''\dot{V}^2\Delta^2}, \qquad F_4 = dC_3, \qquad H_3 = dB_2\,, \end{aligned}$$

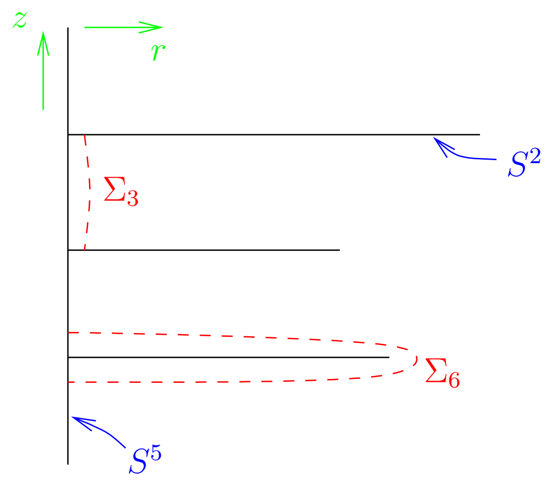

Figure 1: An example electrostatics configuration. The conducting disks are the horizontal solid lines. Fibred above the rz-plane are an S^2 and an S^5. The size of the S^2 shrinks on the disks, whereas the S^5 shrinks on the z axis. The dashed lines indicate topological 3- and 6-cycles in the geometry.

$$\begin{aligned} C_1 &= -\frac{2\dot{V}'\dot{V}}{\ddot{V}-2\dot{V}}dt, \qquad C_3 = -4\frac{\dot{V}^2V''}{\Delta}dt\wedge d^2\Omega\,, \quad B_2 = 2\left(\frac{\dot{V}\dot{V}'}{\Delta}+z\right)d^2\Omega\,, \\ \Delta &\equiv (\ddot{V}-2\dot{V})V''-(\dot{V}')^2, \end{aligned} \tag{2.1}$$

where V is the electrostatics potential, and dots and primes indicate derivatives with respect to $\log r$ and z, respectively. To avoid conical singularities in (2.1), when the size of the S^2 or S^5 shrinks, requires that either V is regular at $r = 0$ (S^5 shrinks) or $\partial_r V = 0$ (S^2 shrinks). Different supergravity solutions can, therefore, be specified by inserting some conducting disks of various radii R_i at positions z_i. See figure 1. Inserting a disk will create separate two regions on the z-axis on which the S^5 shrinks and will therefore mean adding a non contractible 6-cycle, which will carry NS5-brane flux. Similarly, the region between two disks, on which the S^2 shrinks, will be a non-contractible 3-cycle carrying D2-brane flux.

Two additional constraints on the electrostatics solution come from ensuring that all of the metric components are positive definite and that the transformation to these coordinates is well defined. Positive definiteness requires that the electrostatics potential takes a definite asymptotic form, and the coordinate transformation requires that the charge density vanishes at the edge of each disk.

We will now describe a method for solving the electrostatics problems for generic configurations.

2.1 General solutions dual to SYM on $R \times S^2$

One of the field theories for which Lin and Maldacena found the corresponding electrostatics configurations is maximally supersymmetric SU(N) Yang-Mills theory on $R \times S^2$. This theory is related to $\mathcal{N} = 4$ SYM on $R \times S^3$ by dimensionally reducing that theory on

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the Hopf fibre of S^3 [2]. The field content is similar to $\mathcal{N} = 4$, however the theory on $R \times S^2$ admits vacuum configurations with non-trivial Φ, the scalar field resulting from the dimensional reduction. The vacua of this theory are parametrized by a set of integers where $\Phi = \mathrm{diag}(n_1, n_2, \ldots, n_N)$ [8]. General discussion of the relations among the vacua in the SU(2|4) symmetric field theories can be found in [9, 10].

The set of electrostatics configurations in question are given by an arbitrary set of positively charged conducting disks. Each disk will be associated with some D2 brane charge, so it is natural to think of the supergravity solutions as being dual to vacua of maximally supersymmetric Yang-Mills on $R \times S^2$ [2]. The integers in the vacuum configuration for Φ are related to positions of the disks by $z = n\pi/2$, and the number of units of charge on the disk is related to the number of times that each integer appears by $Q = \pi^2 N/8$ [2].

The solutions to these electrostatics problems have been given in some specific cases [2, 5, 11]. For example the limit that the disks are very large, or only the geometry near the tip of a disk is of interest, the problem becomes two dimensional and it is possible to treat it with conformal mapping [2, 5, 6, 11]. In the case that there is a single disk it is possible to find an exact solution [2, 5]. For two equally sized disks a formal solution can be found [5, 11]. We will show that the techniques of [5, 11, 12] to solve the electrostatics problem for two identical disks can be extended to more arbitrary disk configurations. We will discuss how these more general solutions may be found using these techniques.

Consider the case of a collection of k charged conducting disks in the case of maximally supersymmetric Yang-Mills theory on $R \times S^2$. This problem is similar to the one for two disks considered in [11], however we will allow the disks here to sit at arbitrary positions, d_i and have arbitrary sizes, R_i. We can take the potential to be

$$V = W_0 \left(r^2 - 2z^2 + \sum_i \phi_i(r, z) \right) , \tag{2.2}$$

where the first two terms ensure the correct asymptotic conditions, and the third is an asymptotically vanishing contribution that comes from the charges on the disks. It takes the form

$$\phi_i(r, z) = \int_0^\infty \frac{du}{u} J_0(ru) A_i(u) e^{-u|z-d_i|} . \tag{2.3}$$

Each function A_i will be shown to determine the charge density on the i^{th} disk. To fix the form of these functions we impose the conducting boundary conditions on the disks. In particular, if the disks are held at fixed potentials Δ_i, then we will find a set of dual integral equations similar in form to those in [5, 11, 12]. The conditions at the i^{th} disk are that for $r < R_i$

$$\int_0^\infty \frac{du}{u} J_0(ur) \left[A_i(u) + \sum_{j \neq i} A_j(u) e^{-u|d_j - d_i|} \right] = \Delta_i + 2d_i^2 - r^2 , \tag{2.4}$$

and for $r > R_i$

$$\int_0^\infty du J_0(ur) A_i(u) = 0 . \tag{2.5}$$

We can make the ansatz

$$A_i(u) = \frac{2u}{\pi} \int_0^{R_i} dt\, \cos(ut) f_i(t)\,, \tag{2.6}$$

so that the conditions in (2.5) are automatically satisfied, and the conditions (2.4) become

$$f_i(r) + \sum_{j \neq i} \int_0^{R_j} dx\, \bar{K}_{ij}(x,r) f_j(x) = g_i(r)\,, \tag{2.7}$$

where

$$\bar{K}_{ij}(x,r) = \frac{|d_i - d_j|}{\pi} \left[\frac{1}{(x+r)^2 + |d_i - d_j|^2} + \frac{1}{(x-r)^2 + |d_i - d_j|^2} \right], \tag{2.8}$$

$g_i(r) = \beta_i - 2r^2$, and $\beta_i = \Delta_i + 2d_i^2$. Since the g_i are all symmetric functions, it is simpler to take the system as

$$f_i(r) + \sum_{j \neq i} \int_{-R_j}^{R_j} dx\, K_{ij}(x,r) f_j(x) = g_i(r)\,, \tag{2.9}$$

where

$$K_{ij}(x,r) = \frac{1}{\pi} \frac{|d_i - d_j|}{(x-r)^2 + |d_i - d_j|^2}\,. \tag{2.10}$$

It is straightforward to show that the charge densities on the disks are given in terms of the f_i as

$$\sigma_i(r) = \frac{W_0}{\pi^2} \left[\frac{f_i(R_i)}{\sqrt{R_i^2 - r^2}} - \int_r^{R_i} du \frac{f_i'(u)}{\sqrt{u^2 - r^2}} \right], \tag{2.11}$$

and that the total charges are

$$Q_i = \frac{W_0}{\pi} \int_{-R_i}^{R_i} du\, f_i(u)\,. \tag{2.12}$$

To find the f_i we must solve the set of linear equations in (2.9), schematically this takes the form

$$\begin{pmatrix} 1 & K_{12} & \cdots \\ K_{21} & 1 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} \beta_1 - 2r^2 \\ \beta_2 - 2r^2 \\ \vdots \end{pmatrix}. \tag{2.13}$$

Due to the complicated form of the kernels K_{ij} this system is not easy to solve analytically, however, it is straightforward to solve it numerically using the Nyström method (see, e.g. [13]). This consists of discretizing the interval and solving the resulting linear system. An additional set of constraints comes from ensuring that the charge densities vanish at the edges of the disks. This amounts to enforcing that $f_i(R_i) = 0$. We define $f_i^{(j)}$ as the set of solutions to (2.9) with $g_i(r) = \delta_i^j$, where if there are N disks $j = 1, \ldots, N$, and $f_i^{(0)}$ as the set with $g_i(r) = 2r^2$. The condition that the charge density vanishes at the edge of the disk is then that

$$\sum_j f_i^{(j)}(R_i)\beta_j = f_i^{(0)}(R_i)\,. \tag{2.14}$$

There will be a unique solution for β_i if $\det(f_i^{(j)}(R_i)) \neq 0$. The full solution f_i is then

$$f_i(r) = -f_i^{(0)}(r) + \sum_j \beta_j f_i^{(j)}(r)\,, \tag{2.15}$$

with the potentials ϕ_i given by

$$\phi_i = \int_{-R_i}^{R_i} dt\, G_i(r,z,t) f_i(t)\,, \tag{2.16}$$

where

$$G_i(r,z,t) = \frac{1}{\pi}\frac{1}{\sqrt{(|z-d_i|+it)^2+r^2}}\,. \tag{2.17}$$

We have therefore reduced the electrostatics problem to a very simple linear system. In the case that there are only two disks, the problem is very straightforward and solution has been used to understand the relationship between SYM on $R\times S^2$ and little string theory [11].

2.2 General solutions dual to the PWMM

Another theory for which Lin and Maldacena found the corresponding electrostatics configurations is the plane wave matrix model [14]. The plane wave matrix model can be found by a consistent truncation of $\mathcal{N}=4$ SYM on $R\times S^3$ to the set of constant modes on the sphere [15]. The vacua of matrix model are given by the scalars that come from the former $\mathcal{N}=4$ gauge field taking values in a representation of SU(2) [14]. Lin and Maldacena [2] associated these vacua with configurations of charged conducting disks above an infinite conducting plane.

The method of solution is very similar to the case above. For the sake of brevity we will give the final solution. We will write the potential as

$$V = V_0\left(r^2 z - \frac{2}{3}z^3 + \sum_i \phi_i(r,z)\right)\,, \tag{2.18}$$

where the first two terms are the background field and the ϕ_i arise from the charged disks as

$$\phi_i(r,z) = \int_{-R_i}^{R_i} dt\, G_i(r,z,t) f_i(t)\,. \tag{2.19}$$

The Green function is

$$G_i(r,z,t) = \frac{1}{\pi}\Big(\frac{1}{\sqrt{(|z-d_i|+it)^2+r^2}} - \frac{1}{\sqrt{(|z+d_i|+it)^2+r^2}}\Big)\,, \tag{2.20}$$

and f_i is a solution of the integral equation

$$f_i(r) + \sum_j \int_{-R_j}^{R_j} dx\, K_{ij}(r,x) f_j(x) = g_i(r)\,, \tag{2.21}$$

in which the kernel is given by

$$K_{ij}(x,r) = \frac{1}{\pi}\left[\frac{|d_i - d_j|}{(x-r)^2 + |d_i - d_j|^2} - \frac{|d_i + d_j|}{(x-r)^2 + |d_i + d_j|^2}\right], \tag{2.22}$$

and $g_i(r) = \beta_i - 2d_i r^2$, where $\beta_i = \Delta_i + \frac{2}{3}d_i^3$. The differences between this solution and the one presented in section 2.1 arise from the presence of the infinite conducting plane that, via the method of images, implies the presence of oppositely charged conducting disks below the image plane. However, the conditions on the charges on the disks (2.11),(2.12), still hold, so the requirement that the charge density vanishes at the edge of the disk is that $f_i(R_i) = 0$. We may again consider solutions to (2.21) in which $g_i(r) = \delta_i^j$, which we will call $f_i^{(j)}$, and $f_i^{(0)}$ for which $g_i(r) = 2d_i r^2$. The condition that the charge density vanishes at the edge of each disk is again (2.14), and f_i will be given by (2.15).

As in the case of SYM on $R \times S^2$, the electrostatics problem has been reduced to a very simple linear system. We will now show how we can solve this system to study instantons at strong coupling on the field theory side using this method.

3. Instanton calculations

Recently, Lin [6] has considered tunnelling between vacua in the plane wave matrix model and in maximally supersymmetric Yang-Mills on $R \times S^2$. It is possible to study this on both the gauge theory and gravity sides. In the gauge theory case, this can be approached by directly studying the instanton solutions [7]. Lin [6] has also shown that it is possible to introduce a superpotential that gives a bound for the instanton action according to

$$S_{\text{inst}} = -\frac{1}{g^2}\Delta W\,. \tag{3.1}$$

Lin [6] further studied this on the gravity side. Explicit answers were given in the case that the disks in the corresponding electrostatics problem were small and could be approximated by point charges. Moreover, Lin [6] gave a prescription for how the instanton action could be expressed in terms of the electrostatics potential in more general cases, but was not able to give explicit expressions.

For completeness, we will briefly review Lin's prescription for finding the instanton action from the gravity side [6], and then we will demonstrate the use of the techniques developed above to calculate the instanton action for some non-trivial electrostatics configurations.

3.1 Instantons on the gravity side

Since Lin and Maldacena [2] have found electrostatics configurations corresponding to SU(2|4) symmetric field theory vacua, it is interesting to understand how instantons in the field theories can be described in the gravity picture. Lin [6] has studied this question by first considering vacua in the field theories for which the electrostatics configurations do not differ drastically. These instantons can be addressed by calculating the action for a Euclidean D2-brane wrapping a non-contractible Σ_3 in the geometry. As discussed in [2],

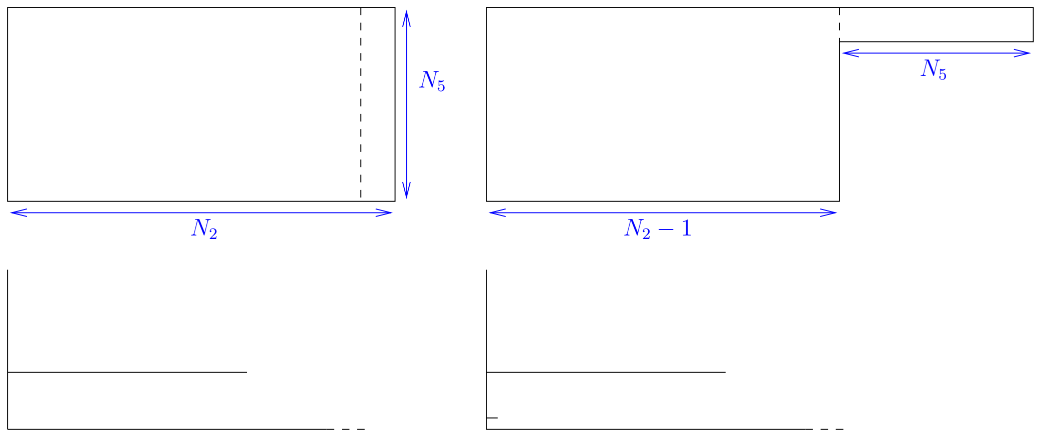

Figure 2: The Young diagrams associated with the initial and final vacua in the plane wave matrix model, and the electrostatics problems for the dual supergravity solutions.

since the brane will be wrapping a cycle carrying some N_5 units of flux, it should have N_5 D0-branes ending on it, and therefore describe the creation of N_5 D0-branes in the throat, see figure 2.

Using the mapping to an electrostatics configuration, the action for the Euclidean D2-brane can be expressed in terms of the solution to the electrostatics problem. Consider the case of a charged conducting disk at a position z_0 above an infinite conducting plane, as shown in figure 2. Lin [6] has shown that the action for such a configuration takes the form

$$S_E = -\frac{2}{\pi}[V(z_0) - V(0) - z_0 V'(0)]\,, \tag{3.2}$$

where V is the electrostatics potential evaluated along $r = 0$, and prime denotes differentiation with respect to z. This expression, however, is proportional to the change in energy of the electrostatics configuration, $S_E = 8\Delta U/\pi^3$. This led Lin [6] to identify the superpotential at strong coupling as

$$W \equiv -\frac{8g^2}{\pi^3}U = -\frac{16g^2}{\pi^3}\sum_i Q_i V_i\,, \tag{3.3}$$

where U is the energy of the electrostatics configuration.

In the case that the disks are small relative to their separation, which is at weak coupling in the gauge theory, the superpotential is given by the energy of a system of point charges. Using the prescription (3.3), Lin found [6]

$$W = \frac{1}{3}\sum_i N_2^{(i)} N_5^{(i)3}\,, \tag{3.4}$$

in perfect agreement with the weak coupling gauge theory results [6, 7]. In the case that the coupling is not weak, the charges arrange to form extended disks. We will find the superpotential at strong coupling by solving the electrostatics problem for a set of extended disks.

3.2 Instantons in the PWMM

In this section we will consider instantons in the simplest non trivial electrostatics configuration, that of a single conducting disk carrying $\pi^2 N_2/8$ units of charge at a distance $\pi N_5/2$ above a conducting plane. This corresponds to a field theory vacuum with N_2 copies of the N_5 dimensional representation. We will determine the superpotential for arbitrary N_2 and N_5, and calculate the action for Euclidean D2-brane wrapping the non-contractible Σ_3.

In the case of small changes to the background mediated by the Euclidean D2-brane, this will compute the action between a vacuum of the PWMM with N_2 copies of the N_5 dimensional representation, and a vacuum with $N_2 - 1$ copies of the N_5 dimensional representation and N_5 copies of the trivial representation. See figure 2.

The electrostatics problem in this case can be approached using the technique outlined in section 2.2 applied to the case of a single disk above the conducting plane. We consider a disk of radius R, at a distance d from the plane, which is a generalization of the approach in [5]. The solution to the electrostatics problem will be

$$V = V_0 \left(r^2 z - \frac{2}{3} z^3 + \phi \right) , \tag{3.5}$$

where ϕ is

$$\phi(r,z) = \int_{-R}^{R} dt\, G(r,z,t) f(t) , \tag{3.6}$$

with Green function

$$G(r,z,t) = \frac{1}{\pi} \left(\frac{1}{\sqrt{(|z-d|^2 + it)^2 + r^2)}} - \frac{1}{\sqrt{(|z+d|^2 + it)^2 + r^2)}} \right) . \tag{3.7}$$

Here f satisfies the integral equation

$$f(r) + \int_{-R}^{R} dx\, K(r,x) f(x) = g(r) , \tag{3.8}$$

with $g(r) = \beta - 2dr^2$, $\beta = \Delta + \frac{2}{3} d^3$, and kernel

$$K(r,x) = -\frac{1}{\pi} \frac{2d}{(x-r)^2 + 4d^2} . \tag{3.9}$$

We will solve the problem by finding a numerical solution to the integral equation (3.8).

Solving this integral equation is straightforward. As a check on our numerical results, we ensured that the asymptotic form for the superpotential in the limit that the number of units of charge on the disks was small is given by (3.4). Indeed, we found that for $N_5 \gg \lambda^{\frac{1}{3}} \gg 1$ [1]

$$W \approx \frac{1}{3} N_2 N_5^3 + a \sqrt{\lambda} N_2 N_5^{\frac{3}{2}} , \tag{3.10}$$

where the numerical constant $a \approx 1.4$.

[1] Here $\lambda \equiv g^2 N_2$, where g is the Yang-Mills coupling of the matrix model.

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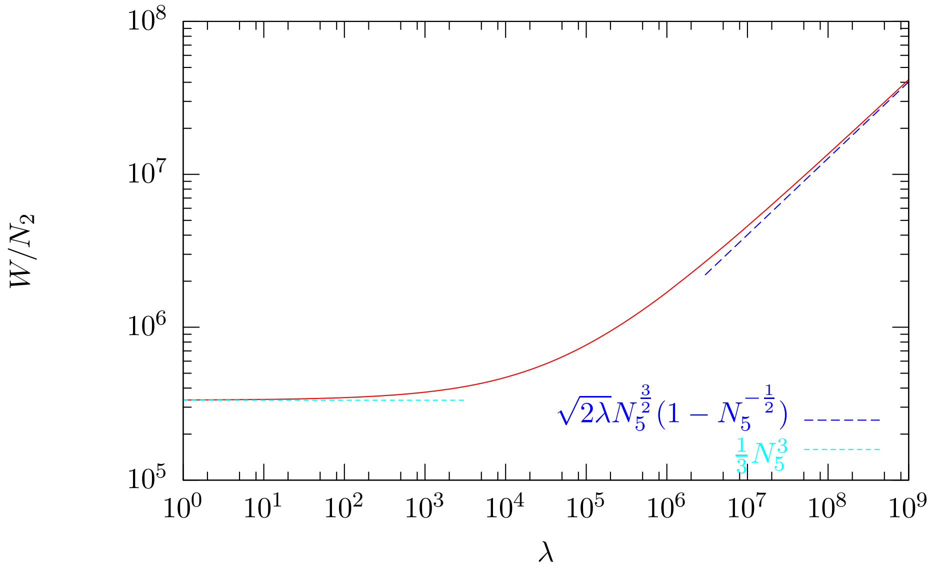

Figure 3: The superpotential for $N_5 = 100$. The dashed lines indicate the asymptotic values for small and large λ compared to N_5.

The solution of the electrostatics problem when the disks are large gives the superpotential at strong coupling. A plot of the superpotential for $N_5 = 100$ is given in figure 3. It is possible to extract the asymptotic form for the superpotential in the limit that $\lambda^{\frac{1}{4}} \gg N_5 \gg 1$. We find the result

$$W \approx b\sqrt{\lambda}N_2N_5^{\frac{3}{2}}(1-N_5^{-\frac{1}{2}}) + c\lambda^{\frac{1}{4}}N_2N_5^2(1-N_5^{-\frac{3}{4}})\,, \tag{3.11}$$

where the numerical constants $b \approx 1.4142 \approx \sqrt{2}$, and $c \approx 0.7$. In defining the superpotential, there was the freedom to choose an overall constant factor. Here we have defined the superpotential to be zero for the vacua given by N_2N_5 copies of the trivial representation.

We can also use the electrostatics solution to determine the action according to (3.2) for the instanton shown in figure 2. A plot of the result for $N_5 = 100$ is shown in figure 4. When λ is small compared to N_5,

$$S_E \approx \frac{1}{3}\frac{N_2N_5^3}{\lambda}\,, \tag{3.12}$$

and it falls off faster than any power of λ when λ is large. When λ is small compared to N_5, the potential on the disk in the electrostatics problem is negative, which is due to the form of the background potential. As the size of the disk increases the potential on the disk increases. The instanton action according to (3.2) begins to fall off from the behaviour at weak coupling near where the potential on the disk crosses zero. It is sensible that it should vanish when the coupling is infinite, since in that case we would expect the electric field to become constant between the disks near the origin, and so the potential difference and dipole contributions should cancel out. Let us briefly mention when these results should be valid. The euclidean brane approximation will be valid when the number of units of charge

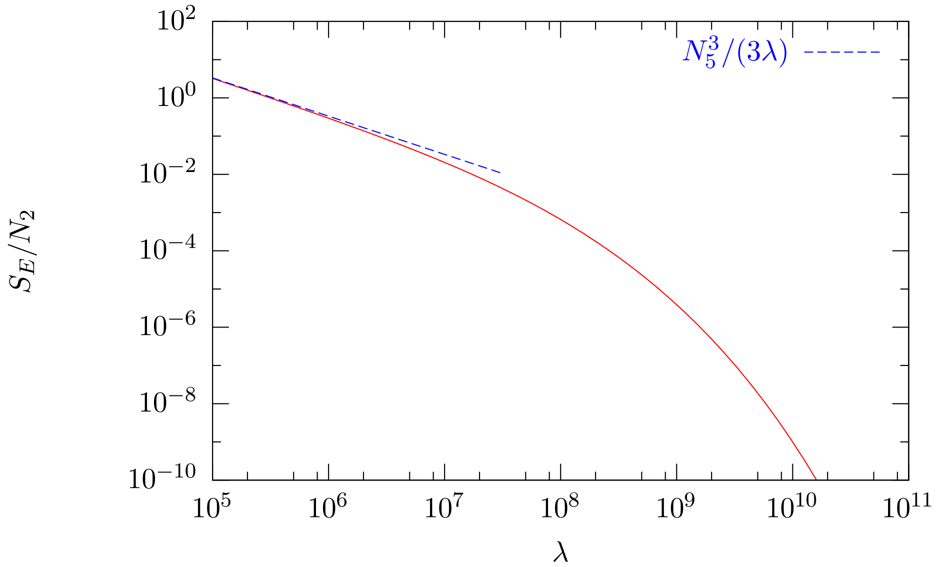

Figure 4: The action for a euclidean D2-brane for the situation shown in figure 2. Here $N_5 = 100$. The dashed line shows the asymptotic behaviour for λ small compared to N_5.

on the disk is large, the potential from the dipole at the origin is small at the surface of the disk, and the curvature is small. Therefore λ, N_2 and N_5 must all be large.

It would be helpful to better understand the scaling behaviour that was found in (3.10) and (3.11). We have taken the normalization of the superpotential thus far to allow for direct comparison of vacua with N_2 copies of the N_5 dimensional representation to ones with $N_2 N_5$ copies of the trivial representation. The asymptotic behaviours we found in (3.10) and (3.11) using this prescription have some interesting features. In particular, at a fixed order in λ, we found that the coefficients for the subleading terms in N_5 are one (i.e. the factors of $(1 - N_5^{-\alpha})$, where α is some positive number). The reason for this is as follows. If we took the superpotential to be normalized to zero for the empty background instead, it would have been advantageous to take out a further scaling factor of $N_5^3 \sim d^3$ in (3.5). In that case, the potential would be of the form $V = V_0 N_5^3 \bar{V}(R/d)$. Likewise, the charge on the disk would then have the form $Q = V_0 N_5^4 \bar{q}(R/d)$. Since the total charge is proportional to N_2, we must have that

$$Q \sim N_2 \sim \frac{N_5^4}{g^2} \bar{q}(R/d)\,, \tag{3.13}$$

or

$$\bar{q}(R/d) \sim \frac{g^2 N_2}{N_5^4} = \frac{\lambda}{N_5^4}\,. \tag{3.14}$$

We see, then, that functions of R/d in the scaled electrostatics problem can only depend on the combination of gauge theory parameters λ/N_5^4. Therefore the superpotential with this alternative normalization must have the form

$$\bar{W}(\lambda, N_2, N_5) = N_2 N_5^3 \bar{w}\left(\frac{\lambda}{N_5^4}\right). \tag{3.15}$$

Our numerical results confirm this. We find the following asymptotic behaviour for $\bar{w}$:

$$\bar{w}(x) \approx \frac{1}{3}\left(1 - \frac{10}{3}x^{\frac{2}{3}}\right), \qquad x \ll 1, \tag{3.16}$$
$$\bar{w}(x) \approx -\sqrt{2}x^{\frac{1}{2}} + 0.7x^{\frac{1}{4}}, \qquad x \gg 1.$$

The superpotential with the original normalization is given in terms of $\bar{W}$ by

$$W(\lambda, N_2, N_5) = \bar{W}(\lambda, N_2, N_5) - \bar{W}(\lambda N_5, N_2 N_5, 1). \tag{3.17}$$

If we combine the contributions from the asymptotic behaviour of each of the terms in this expression that come from the behaviour found in (3.16), then we will recover the asymptotic behaviour that was found in (3.10) and (3.11). The factors of $(1 - N_5^{-\alpha})$ occur as a result of those combinations.

4. Discussion

In this paper we have given a prescription for finding the supergravity solutions dual to general vacua of the plane wave matrix model and maximally supersymmetric Yang-Mills on $R \times S^2$ by using the mapping of Lin and Maldacena [2] on to axisymmetric electrostatics problems.

The prescription extends the technique developed in [5, 11] to arbitrary electrostatics configurations. The electrostatics problems are reduced to a set of integral equations that can be solved quite straightforwardly using the Nyström method.

We have shown that an application of the prescription to a specific case can be used to study instantons at strong coupling in the plane wave matrix model. In particular we found that the instanton action for a transition between a vacuum described by N_2 copies of the N_5 dimensional representation and one by $N_2 - 1$ copies of the N_5 dimensional representation and N_5 copies of the trivial representation falls off faster than any power of λ at strong coupling (see figure 4). We also found that at strong coupling the superpotential for a vacuum with N_2 copies of the N_5 dimensional representation behaves like $\sqrt{2\lambda}N_2 N_5^{3/2}$, when N_5 is large (see figure 3). This demonstrates that the techniques developed above are useful for obtaining strong coupling results in the field theory.

One question that would be interesting to address using this method is to calculate the superpotential explicitly for more general electrostatics configurations. For example, studying the vacuum of maximally supersymmetric Yang-Mills theory on $R \times S^2$ in which $\Phi = \text{diag}(n, \ldots, n, -n, \ldots, -n)$, a similar scaling could be applied as was done for the plane wave matrix model. In that case we would expect that the corrections to the weak coupling results given by Lin [6] would depend on the parameter λ/n^3. It would certainly be interesting to study that case in detail, as well as other more general vacua.

One open question is to prove that requiring the charge density to vanish at the edge of each disk implies that there is a unique solution to the electrostatics problem. The condition for a unique solution to exist is given above by requiring $\det(f_i^{(j)}(R_i)) \neq 0$. We have not been able to prove that this is true in general. It would be interesting to do so.

Finally, it would be very interesting to use this method for finding the dual geometry to study other strong coupling phenomena on the gauge theory side.

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